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SINGULARITY OF TOPOLOGICAL CONJUGACIES BETWEEN CERTAIN UNIMODAL MAPS OF THE INTERVAL†

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ABSTRACT
Let τ_1 and τ_2 be unimodal maps of the interval $[0,1]$. Let ψ be a topological conjugacy between τ_1 and τ_2. Sufficient conditions are given which guarantee that ψ is singular.

1. Introduction

Let $\{\tau_\lambda : \lambda > 1\}$ be a family of maps from $[0,1]$ onto $[0,1]$ defined by

$$\tau_\lambda(x) = \begin{cases} \lambda x, & 0 \leqq x \leqq \dfrac{1}{\lambda}, \\ \left(\dfrac{\lambda}{1-\lambda}\right)(x-1), & \dfrac{1}{\lambda} \leqq x \leqq 1. \end{cases}$$

Let $\tau : [0,1] \to [0,1]$ be continuous and let it possess a unique turning point c. Under certain further restrictions on τ, we show that the homeomorphism $\psi : [0,1] \to [0,1]$, which renders τ_λ and τ topologically conjugate, is singular.

2. Main results

Let $\sigma : [0,1] \to [0,1]$ be any piecewise monotonic continuous map with unique maximum, satisfying $\sigma(0) = 0 = \sigma(1)$. To every such map we can associate a symmetry map $\beta : [0,1] \to [0,1]$ such that $\beta(x) = y$ if and only if $\sigma(x) = \sigma(y)$, $\beta \neq$ identity.

Let τ_λ, $\lambda > 1$, be as above. The symmetry map of τ_λ is given by

$$\beta_\lambda(x) = \begin{cases} 1 + (1-\lambda)x, & 0 \leqq x \leqq \dfrac{1}{\lambda}, \\ \dfrac{x-1}{1-\lambda}, & \dfrac{1}{\lambda} \leqq x \leqq 1. \end{cases}$$

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Let Q denote the class of continuous maps $\tau : [0,1] \to [0,1]$, not necessarily expanding, which satisfy:

(i) τ has a unique turning point c,

(ii) τ is increasing and C^1 on $(0,c)$ and decreasing and C^1 on $(c,1)$,

(iii) $\tau(0)=\tau(1)=0$, $\tau(c)=1$,

(iv) $1<\lambda_2 \equiv \inf_{(0,c)} |\tau'(x)|$.

LEMMA 1. *Let $\tau \in Q$ and define $M_2 = \sup_{(0,c)} |\beta'(x)|$, where β is the symmetry map for τ. If $1<\lambda<\lambda_2$, then*

$$a = \log\left(\frac{\lambda-1}{M_2}\right) \Big/ \log\left(\frac{\lambda}{\lambda_2}\right) > 1.$$

PROOF. Clearly $\lambda_2 \leqq 1/c$. Since $\beta(0)=1$ and $\beta(c)=c$,

$$M_2 \geqq 1/c - 1 \geqq \lambda_2 - 1.$$

Thus,

$$a = \log\left(\frac{M_2}{\lambda-1}\right) \Big/ \log\left(\frac{\lambda_2}{\lambda}\right) > \log\left(\frac{\lambda_2-1}{\lambda-1}\right) \Big/ \log\left(\frac{\lambda_2}{\lambda}\right) > 1. \qquad \text{Q.E.D.}$$

THEOREM 1. *Let $\tau \in Q$. Assume*

$$\left(\frac{\lambda}{\lambda_2}\right)^{\lambda} < \left(\frac{\lambda-1}{M_2}\right)^{\lambda-1} \tag{1}$$

for some $\lambda>1$, where $\lambda_2 \neq \lambda$. Then any monotonic semi-conjugacy ψ from τ_λ onto τ (i.e. $\psi : [0,1]\to[0,1]$ is monotonic and satisfies $\psi \circ \tau_\lambda = \tau \circ \psi$) is singular.

PROOF. We begin by specifying a certain countable dense subset of $[0,1/\lambda]$ on which ψ can be determined exactly. To that end, let D denote the set of inverse images under τ_λ of the turning point $1/\lambda$ having the form

$$\lambda^{-n_1} + (1-\lambda)^{-(n_1+n_2)} + \cdots + (1-\lambda)^{k-1}\lambda^{-(n_1+\cdots+n_k)},$$

where $\{n_1, n_2, \cdots, n_k\}$ is any finite sequence of positive integers. Note that $(1-\lambda)<0$. We begin by showing that D is dense. Given any $x \in [0,1/\lambda]-D$, choose n_1 such that

$$\lambda^{-(n_1+1)} < x < \lambda^{-n_1}$$

and set $x_1 = \lambda^{-n_1}$. Then,

$$x_1 - x < \left(\frac{\lambda-1}{\lambda}\right)\lambda^{-n_1} \quad \text{or} \quad x_1 - \left(\frac{\lambda-1}{\lambda}\right)\lambda^{-n_1} < x.$$

Since $(\lambda - 1)/\lambda^n \to 0$, we can find an $n_2 \geqq 1$ such that

$$\lambda^{-n_1} + (1-\lambda)\lambda^{-(n_1+n_2)} < x < \lambda^{-n_1} + (1-\lambda)\lambda^{-(n_1+n_2+1)}.$$

Letting $x_2 = \lambda^{-n_1} + (1-\lambda)\lambda^{-(n_1+n_2)}$, we get

$$x < x_2 + \frac{(\lambda - 1)^2}{\lambda}\lambda^{-(n_1+n_2)}.$$

Thus, we can find $n_3 \geqq 1$ such that

$$\lambda^{-n_1} + (1-\lambda)^{-(n_1+n_2)} + (1-\lambda)^2\lambda^{-(n_1+n_2+n_3+1)} < x$$
$$< \lambda^{-n_1} + (1-\lambda)\lambda^{-(n_1+n_2)} + (1-\lambda)^2\lambda^{-(n_1+n_2+n_3)}.$$

Then

$$x_3 - \frac{(\lambda - 1)^3}{3}\lambda^{-(n_1+n_2+n_3)} < x$$

where $x_3 = \lambda^{-n_1} + (1-\lambda)\lambda^{-(n_1+n_2)} + (1-\lambda)^2\lambda^{-(n_1+n_2+n_3)}$. We continue in this way to define a sequence of integers $n_1, n_2, \cdots$ and a sequence of points $\{x_k\}_{k=1}^{\infty} \in D$ defined by

$$x_k = \sum_{i=1}^{k}(1-\lambda)^{i-1}\lambda^{-(n_1+n_2+\cdots+n_i)}.$$

Since

$$|x - x_k| < \frac{(\lambda - 1)^k}{\lambda^{k+1}} \to 0 \qquad \text{as } k \to \infty,$$

the sequence $\{x_k\}_{k=1}^{\infty}$ converges to x.

We next consider how points in D behave under the semiconjugacy ψ. The conjugacy condition may be written as follows:

$$\psi(x) = \begin{cases} \tau_{2,1}^{-1}\psi(\lambda x), & 0 \leqq x \leqq \dfrac{1}{\lambda}, \\ \tau_{2,1}^{-1}\psi\left(\dfrac{\lambda}{\lambda - 1}(1-x)\right), & \dfrac{1}{\lambda} \leqq x \leqq 1, \end{cases} \tag{2}$$

where $\tau_{2,1} = \tau_2|_{[0,c]}$ and $\tau_{2,2} = \tau_2|_{[c,1]}$. Equivalently, we have

$$\begin{cases} \psi(x/\lambda) = \tau_{2,1}^{-1}(\psi(x)), & 0 \leqq x \leqq 1, \\ \psi(1-(\lambda-1)x) = \tau_{2,2}^{-1}(\tau_{2,1}(\psi(x))) = \beta\psi(x), & 0 \leqq x \leqq 1/\lambda. \end{cases} \tag{3}$$

It follows that

$$\psi\left(\frac{1}{\lambda^n}\right) = \tau_{2,1}^{-n}(1), \qquad \psi\left(1 - \frac{\lambda - 1}{\lambda^n}\right) = \beta\tau_{2,1}^{-n}(1),$$

and in general,

$$\psi\left(\frac{1}{\lambda^{n_1}}+\frac{(1-\lambda)}{\lambda^{n_1+n_2}}+\cdots+\frac{(1-\lambda)^{k-1}}{\lambda^{n_1+\cdots+n_k}}\right)=\tau_{2,1}^{-n_1}\circ\beta\circ\tau_{2,1}^{-n_2}\circ\beta\circ\cdots\circ\beta\circ\tau_{2,1}^{-n_k}(1).$$

Therefore, given any $x\in[0,1/\lambda]-D$ with approximating sequence $\{x_k\}_{k=1}^{\infty}$ given by the sequence of integers $\{n_k\}_{k=1}^{\infty}$, we have

$$|\psi(x_k)-\psi(x_{k-1})|<(1/\lambda_2)^{n_1+n_2+\cdots+n_{k-1}}M_2^{k-2}|\beta\tau_{2,1}^{-n_k}(1)-1|.$$

But

$$|\beta\tau_{2,1}^{-n_k}(1)-1|\leqq M_2\tau_{2,1}^{-n_k}(1)\leqq M_2(1/\lambda_2)^{n_k},$$

since $\beta(0)=1$. Thus,

$$|\psi(x_k)-\psi(x_{k-1})|<M_2^{k-1}(1/\lambda_2)^{n_1+\cdots+n_k}.$$

Note also that

$$|x_k-x_{k-1}|=\frac{(\lambda-1)^{k-1}}{\lambda^{n_1+\cdots+n_k}}.$$

Hence,

$$\left|\frac{\psi(x_k)-\psi(x_{k-1})}{x_k-x_{k-1}}\right|\leqq\left(\frac{\lambda}{\lambda_2}\right)^{n_1+\cdots+n_k}\left(\frac{M_2}{\lambda-1}\right)^{k-1}.$$

We will show that

$$\liminf_{k\to\infty}\frac{\psi(x_k)-\psi(x_{k-1})}{x_k-x_{k-1}}=0 \tag{4}$$

for almost every $x\in[0,1/\lambda]$. Since the points x_{k-1} and x_k lie on opposite sides of x, this implies that $\psi'(x)=0$ a.e. on $[0,1/\lambda]$ and hence on $[0,1]$ by (2). Thus ψ is singular.

Let $x\in(0,1/\lambda)-D$, and consider the inequality

$$\left(\frac{\lambda}{\lambda_2}\right)^{n_1+\cdots+n_k}\left(\frac{M_2}{\lambda-1}\right)^{k-1}<\varepsilon. \tag{5}$$

If $\lambda>\lambda_2$, this is equivalent to

$$n_1+\cdots+n_k<ak+b_1 \tag{6a}$$

whereas if $\lambda<\lambda_2$ we get

$$n_1+\cdots+n_k>ak+b_2 \tag{6b}$$

where

$$a = \log\left(\frac{\lambda - 1}{M_2}\right) \Big/ \log\left(\frac{\lambda}{\lambda_2}\right)$$

and b_1, b_2 are constants (depending on ε). We note that $a > 1$: this follows from (1) if $\lambda > \lambda_2$ and from Lemma 1 if $\lambda < \lambda_2$.

If $\alpha < a$ and $n_1 + \cdots + n_k \leqq \alpha k$ is satisfied for arbitrarily large k, so is 6(a) for any b_1. Similarly, if $\alpha > a$ and $n_1 + \cdots + n_k > \alpha k$ holds for infinitely many values of k, so does 6(b) for any b_2.

Let $E = \{x \in (0, 1/\lambda) \mid \psi'(x)$ exists$\}$; we note that E is measurable and $m(E) = 1/\lambda$ (m = Lebesgue measure). Let

$$A_k = A_k(\alpha) = \{x \in (0, 1/\lambda) - D \mid n_1 + \cdots + n_k \leqq \alpha k\},$$

$$A'_k = A'_k(\alpha) = \{x \in (0, 1/\lambda) - D \mid n_1 + \cdots + n_k > \alpha k\},$$

$$E_k = A_k \cap E \quad \text{and} \quad E'_k = A'_k \cap E.$$

Then $m(A_k) = m(E_k)$, $m(A'_k) = m(E'_k)$ and $m(A_k \cup A'_k) = 1/\lambda$ for a fixed α.

If $\lambda > \lambda_2$ and $\alpha < a$ the above implies that $\limsup_k E_k \subset \{x \in E \mid \psi'(x) = 0\}$, whereas if $\lambda < \lambda_2$ and $\alpha > a$ we have

$$\limsup_k E'_k \subset \{x \in E \mid \psi'(x) = 0\}.$$

Thus if we show that

$$m(E_{k_i}) = m(A_{k_i}) \to \begin{cases} 1/\lambda & \text{if } \lambda > \lambda_2 \\ 0 & \text{if } \lambda < \lambda_2 \end{cases}$$

for some sequence $\{k_i\}$ of positive integers and for all rational α near a, the result follows. Let

$$F_k = \{x \in (0, 1/\lambda) - D \mid n_1, \cdots, n_k \text{ in the approximating sequence for } x \text{ are fixed}\}.$$

Then

$$m(F_k) = \frac{1}{\lambda} \frac{(\lambda - 1)^k}{\lambda^{(n_1 + \cdots + n_k)}}$$

and

$$m(A_k) = m\left(\bigcup_{k=1}^{\infty} \{F_k \mid n_1 + \cdots + n_k \leqq \alpha k\}\right)$$

$$= \sum_{\substack{n_1 + \cdots + n_k \leqq \alpha k \\ n_i \geqq 1}} m(F_k)$$

(since the F_k are pairwise disjoint)

$$=\frac{1}{\lambda}\sum_{n=0}^{(\alpha-1)k}\sum_{\substack{n_1+\cdots+n_k=k+n\\ n_i\geqq 1}}\frac{(\lambda-1)^k}{\lambda^{n_1+\cdots+n_k}},$$

where we choose k so that αk is integral.

Now, for a fixed n, the number of distinct k-tuples $(n_1,\cdots,n_k)$ satisfying $n_1+\cdots+n_k=k+n$ is easily seen to be $\binom{k+n-1}{k-1}$. Thus

$$m(A_k)=\frac{1}{\lambda}\sum_{n=0}^{(\alpha-1)k}pp^{k-1}q^n\binom{n+k-1}{k-1},$$

where we have put $p=1-1/\lambda$, $q=1-p=1/\lambda$. Since $p^{k-1}q^n\binom{n+k-1}{k-1}$ is the probability of the event $S_{n+k-1}=k-1$: $k-1$ successes after $n+k-1$ Bernoulli trials with probability p of success, we have

$$m(A_k)=\frac{1}{\lambda}\sum_{n=0}^{(\alpha-1)k}pP(S_{n+k-1}=k-1)$$

$$=\frac{1}{\lambda}\sum_{n=0}^{(\alpha-1)k}P(S_{n+k-1}=k-1\text{ and }S_{n+k}=k)$$

$$=\frac{1}{\lambda}\sum_{n=0}^{(\alpha-1)k}P(\text{first occurrence of }k^{\text{th}}\text{ success is at trial }n+k)$$

$$=\frac{1}{\lambda}P(S_{\alpha k}\geqq k).$$

By the central limit theorem, if

$$I_k(\alpha)=\frac{1}{\lambda}\frac{1}{\sqrt{2\pi}}\int_{\frac{k(1-\alpha p)}{\sqrt{\alpha kpq}}}^{\infty}e^{-x^2/2}dx$$

then

$$\lim_i I_{k_i}=\lim_i m(A_{k_i}),$$

where $k_i\alpha$ are integral.

Assume first $\lambda>\lambda_2$. From (1) we have

$$\left(\frac{\lambda-1}{\lambda}\right)\log\left(\frac{\lambda-1}{M_2}\right)\Big/\log\left(\frac{\lambda}{\lambda_2}\right)>1,$$

i.e., $ap>1$ and so $1-\alpha p<0$ for rational α close to a. Thus

$$\lim_i m(A_{k_i})=\lim_i I_{k_i}=\frac{1}{\lambda}.$$

If $\lambda < \lambda_2$, (1) is equivalent to $ap < 1$ and so $1 - \alpha p > 1$ for α close to a. Thus

$$\lim_i m(A_{k_i}) = \lim_i I_{k_i} = 0. \qquad \text{Q.E.D.}$$

To show that Theorem 1 has content, we establish

LEMMA 2. *If $a > b > 1$ then*

$$\left(\frac{a-1}{b-1}\right)^{a-1} > \left(\frac{a}{b}\right)^a > \left(\frac{a}{b}\right)^b > \left(\frac{a-1}{b-1}\right)^{b-1}$$

PROOF. Put

$$f(x) = \left(\frac{a-x}{b-x}\right)^{c-x}.$$

We must show that if $c = a$, $f(0) < f(1)$ and if $c = b$, $f(0) > f(1)$ (the middle inequality is trivial). By elementary calculus,

$$\frac{f'}{f} = \frac{(c-x)(a-b)}{(a-x)(b-x)} - \log\left(\frac{a-x}{b-x}\right); \quad x < b.$$

If $c = a$,

$$\frac{f'}{f} = \frac{a-b}{b-x} - \log\left[1 + \frac{a-b}{b-x}\right] > 0.$$

If $c = b$,

$$\frac{f'}{f} = 1 - \frac{b-x}{a-x} - \log\left(\frac{a-x}{b-x}\right) < 0.$$

COROLLARY 1. *If $1 < \lambda_2 \neq \lambda$, then τ_{λ_2} satisfies the conditions of Theorem 1. Moreover, if τ is any map in Q, which differs from some such τ_{λ_2} by a sufficiently C^1-small perturbation, then any monotonic semiconjugacy from τ_λ onto τ is singular.*

PROOF. If $\lambda_2 > \lambda > 1$, we have from Lemma 2 that

$$\left(\frac{\lambda_2}{\lambda}\right)^\lambda > \left(\frac{\lambda_2 - 1}{\lambda - 1}\right)^{\lambda - 1}.$$

Since $M_2 = \lambda_2 - 1$ if $\tau = \tau_{\lambda_2}$, it follows that (1) of Theorem 1 is also satisfied.

If $\lambda > \lambda_2 > 1$ we have from Lemma 2,

$$\left(\frac{\lambda - 1}{\lambda_2 - 1}\right)^{\lambda - 1} > \left(\frac{\lambda}{\lambda_2}\right)^\lambda,$$

which again is (1) of Theorem 1.

Since (1) is an open condition, it is also satisfied by all τ close to τ_{λ_2}.

Q.E.D.

REMARK. By putting $\sigma_2 = \sup_{(0,c)} |\tau'|$ and $m_2 = \inf_{(0,c)} |\beta'|$ one obtains a lower bound for

$$\left| \frac{\psi(x_k) - \psi(x_{k-1})}{x_k - x_{k-1}} \right|.$$

One can then show, using the same arguments as in Lemma 1 and Theorem 1, that if

$$\left(\frac{\lambda}{\sigma_2}\right)^{\lambda} > \left(\frac{\lambda - 1}{m_2}\right)^{\lambda - 1},$$

then

$$\psi'(x) = \infty \qquad \text{a.e.},$$

and conclude from this contradiction that a monotonic semiconjugacy ψ from τ_λ to τ does not exist. However, this result is vacuous, since

$$\left(\frac{\lambda}{\sigma_2}\right)^{\lambda} \leqq \left(\frac{\lambda}{c^{-1}}\right)^{\lambda} \leqq \left(\frac{\lambda - 1}{c^{-1} - 1}\right)^{\lambda - 1} \leqq \left(\frac{\lambda - 1}{m_2}\right)^{\lambda - 1}$$

where c is the turning point of τ. The middle inequality is strict if $\lambda \neq c^{-1}$ by Corollary 1.

EXAMPLE. Condition (1) can be written as

$$M_2 < \frac{(\lambda - 1)}{\lambda^{\lambda/(\lambda - 1)}} \lambda_2^{\lambda/(\lambda - 1)}.$$

Let $\lambda = 2$. Then,

$$M_2 < \tfrac{1}{4} \lambda_2^2.$$

Choose $\lambda_2 = 4$ and $\tau(x) = 4x$ on $[0, \frac{1}{4}]$. Then, since $M_2 \geqq \lambda_2 - 1$, we require: $3 \leqq M_2 < 4$. Let $0 < m < 1$ and let

$$\tau(x) = (\tfrac{4}{3})^m (1 - x)^m, \qquad \tfrac{1}{4} \leqq x \leqq 1.$$

Clearly, τ is continuous on $[0,1]$. Now $\beta(x) = y$ iff $\tau(x) = \tau(y)$, i.e. iff

$$4x = (\tfrac{4}{3})^m (1 - y)^m,$$

or

$$y = 1 - \tfrac{3}{4} 4^{1/m} x^{1/m}.$$

Thus, on differentiating, we obtain:

$$|\beta'(x)| = \left| \frac{dy}{dx} \right| = \tfrac{3}{4} 4^{1/m} \frac{1}{m} x^{1/m-1},$$

and

$$M_2 = \sup_{(0,c)} |\beta'(x)| = \tfrac{3}{4} 4^{1/m} \frac{1}{m} (\tfrac{1}{4})^{1/m-1} = \frac{3}{m}.$$

Hence, $3 \leqq M_2 < 4$ iff $\frac{3}{4} < m \leqq 1$. Thus, for any such m, condition (1) of Theorem 1 holds.

Given any $\lambda_2 > 1$ and $M_2 > \lambda_2 - 1$ we can define a map $\tau \in Q$ as follows. Choose the turning point c such that $1/c < \lambda_2 < M_2 + 1$ and construct a strictly increasing map τ_1 on $[0, c]$ such that $\tau_1(0) = 0$, $\tau_1(c) = 1$ and $\inf |\tau_1'(x)| = \lambda_2$.

Now, it is easy to see that any symmetry map $\beta : [0,1] \to [0,1]$ satisfies: $\beta(0) = 1$, $\beta(c) = c$, $\beta(1) = 0$ and $\beta|_{[c,1]}$ is the reflection of $\beta|_{[0,c]}$ around the line $y = x$. Since $M_2 > 1/c - 1$ and the slope of the line joining points (0,1) and (c, c) is $-(1/c - 1)$, we can construct $\beta|_{[0,c]}$ satisfying

$$M_2 = \sup_{(0,c)} |\beta'(x)| > 1/c - 1.$$

By reflecting around $y = x$, we define β on $[0,1]$.

Thus, so far, we have defined $\tau|_{[0,c]}$ and β. To define $\tau_2 = \tau|_{[c,1]}$, we note that from the definition of β, we have

$$\tau_2(x) = \tau_1(\beta(x)).$$

This defines τ_2. Clearly $\tau_2(c) = 1$, $\tau_2(1) = 0$ and since $\tau_1'(x) > 0$, $\beta'(x) < 0$, $\tau_2' < 0$. Thus τ_2 is monotonic decreasing and hence $\tau \in Q$.

PROPOSITION 1. *Given any $\lambda_2 > 1$ and $M_2 > \lambda_2 - 1$, let $\tau \in Q$ be a map possessing these parameters. Then there exists $\lambda > 1$ such that the topological conjugacy from τ_λ onto τ is singular.*

PROOF. We have only to show that condition (1) is satisfied for some $\lambda > 1$. To see that this is, indeed, the case let $\lambda \downarrow 1$ in (1) to obtain the inequality

$$1/\lambda_2 < 1,$$

since $x^x \to 1$ as $x \to 0$, which is obviously satisfied. Q.E.D.

COROLLARY 2. *Let τ_2 be as in Theorem 1 and let it have injective invariant coordinate [1] (for example, if τ_2 is expanding [1,2]), then τ_λ is topologically conjugate to τ_2 and the homeomorphisms ψ and ψ^{-1} are singular.*

PROOF. That the conjugacy exists and is a homeomorphism follows from arguments in [1, 2, 4]. The above result shows that ψ is singular. It follows that ψ^{-1} must also be singular [5, p. 271, problem 13].

COROLLARY 3. *Any expanding map τ_2 which satisfies the conditions of Theorem 1 admits an uncountable family of invariant ergodic measures which are mutually singular.*

PROOF. Consider the space $\mathscr{C}$ of continuous functions $f:[0,1]\to[0,1]$ satisfying $f(0)=0$, $f(1)=1$ and the transformation $T:\mathscr{C}\to\mathscr{C}$ defined by

$$(T_\lambda f)(x)=\begin{cases}(\tau_{2,1}^{-1}\circ f\circ\tau_\lambda)(x), & 0\leqq x\leqq 1/\lambda,\\ (\tau_{2,2}^{-1}\circ f\circ\tau_\lambda)(x), & 1/\lambda\leqq x\leqq 1.\end{cases}\tag{7}$$

$\mathscr{C}$ is complete in the uniform metric and, since τ_2 is expanding, T_λ is a contraction mapping with unique fixed point ψ_λ. Theorem 1 and Corollary 1 apply to all maps τ_2 with $\lambda>\max\{\lambda_2, M_2+1\}$ or $\lambda<\min\{\lambda_2, M_2+1\}$. Let λ', λ'', $\lambda'\neq\lambda''$ be any two allowable values for λ. Let ψ_1 and ψ_2 be the conjugacies, respectively, which map $\tau_{\lambda'}$, $\tau_{\lambda''}$ onto τ_2. Then

$$m_{\lambda'}(A)=m(\psi_1^{-1}(A))\quad\text{and}\quad m_{\lambda''}(A)=m(\psi_2^{-1}(A))$$

are two continuous ergodic measures invariant under τ_2. Since τ_λ is ergodic with invariant measure m for any $\lambda>1$,

$$m_{\lambda'}[0,c]=1/\lambda'\quad\text{and}\quad m_{\lambda''}[0,c]=1/\lambda'',$$

$m_{\lambda'}$ and $m_{\lambda''}$ are mutually singular. Q.E.D.

3. Some consequences

PROPOSITION 2. *Let τ_λ and τ be as in Theorem 1, and let ψ be the conjugacy which takes τ_λ to τ. Let $\tau_1:[0,1]\to[0,1]$ be any map which is taken to τ_λ by the conjugacy ϕ whose inverse is absolutely continuous. Then the topological conjugacy $\psi\circ\phi$ which takes τ_1 to τ is singular.*

PROOF. Let E be a set of Lebesgue measure 0. Then $B=\phi^{-1}(E)$ has Lebesgue measure 0 since ϕ^{-1} is absolutely continuous. Since ψ is singular there exists a set E, $m(E)=0$, such that $m(\psi(E))>0$. But $E=\phi(B)$, where $m(B)=0$. Hence $m(\psi\circ\phi(B))>0$, and $\psi\circ\phi$ is singular.

EXAMPLE. Let $\tau_1(x)=4x(1-x)$. Then $\phi(x)=(2/\pi)\sin^{-1}(\sqrt{x})$ takes τ_1 to

$$\bar{\tau}(x)=\begin{cases}2x, & 0\leqq x\leqq\frac{1}{2},\\ 2(1-x), & \frac{1}{2}\leqq x\leqq 1.\end{cases}$$

Let τ_λ, $\lambda \neq 2$, play the role of τ in Theorem 1, where ψ, singular, takes $\bar{\tau}$ to τ_λ. Since $\phi(x) = (2/\pi)\sin^{-1}(\sqrt{x})$, ϕ^{-1} is absolutely continuous and $\psi \circ \phi$ taking τ_1 to τ_λ is singular. Hence $\bar{\tau}$ is the only τ_λ map for which the topological conjugacy between τ_1 and τ_λ is absolutely continuous.

In [3], Kowalski studied continuity of absolutely continuous invariant measures for piecewise monotonic expanding maps. In this section we obtain a related result for continuous invariant measures.

For $\lambda > 1$, define the map $T_\lambda : \mathscr{C} \to \mathscr{C}$ by (7). T_λ is a contraction mapping and hence has a unique fixed point $\psi_\lambda \in \mathscr{C}$ for each $\lambda > 1$. Let $\| \quad \|$ denote the uniform norm on $\mathscr{C}$, and let J denote either of the intervals (α, ∞) or $(1, \rho)$, where $\alpha = \max\{\lambda_2, M_2 + 1\}$ and $\rho = \min\{\lambda_2, M_2 + 1\}$.

Lemma 3. *The map $J \to (\mathscr{C}, \| \quad \|)$ defined by $\lambda \to T_\lambda^k f$, for $f \in \mathscr{C}$ fixed, is continuous for all $k \geqq 1$.*

Proof. Since

$$(T^k f)(x) = \begin{cases} \tau_{2,1}^{-1} \circ \cdots \circ \tau_{2,1}^{-1} \circ f \circ \overbrace{\tau_\lambda \circ \cdots \circ \tau_\lambda}^{k \text{ times}}(x), & 0 \leqq x \leqq 1/\lambda, \\ \tau_{2,2}^{-1} \circ \cdots \circ \tau_{2,2}^{-1} \circ f \circ \tau_\lambda \circ \cdots \circ \tau_\lambda(x), & 1/\lambda \leqq x \leqq 1, \end{cases}$$

and τ_λ is continuous with respect to λ, we obtain the result. Q.E.D.

Proposition 3. *The map $J \to (\mathscr{C}, \| \quad \|)$ defined by $\lambda \to \psi_\lambda$ is continuous.*

Proof. Let $\lambda_n \to \lambda$ as $n \to \infty$, where $\{\lambda_n\} \subset J$ and $\lambda \in J$. Then

$$\| \psi_{\lambda_n} - \psi_\lambda \| \leqq \| \psi_{\lambda_n} - T_{\lambda_n}^k \psi_\lambda \| + \| T_{\lambda_n}^k \psi_\lambda - T_\lambda^k \psi_\lambda \| + \| T_\lambda^k \psi_\lambda - \psi_\lambda \|.$$

Since T_{λ_n} is a contraction, $T_{\lambda_n}^k \psi_\lambda \to \psi_{\lambda_n}$ as $k \to \infty$. The second term on the right-hand side tends to 0 as $n \to \infty$ by Lemma 3. The third term is identically 0.

Q.E.D.

Now for each $\lambda \in J$, ψ_λ induces a continuous ergodic measure which is invariant with respect to τ and which is given by $m_\lambda(A) = m(\psi_\lambda^{-1}(A))$. The distribution function associated with m_λ is

$$F_\lambda(x) = m(\psi_\lambda^{-1}[0, x]) = \psi_\lambda^{-1}(x).$$

Proposition 4. *The map $J \to (\mathscr{C}, \| \quad \|)$ defined by $\lambda \to F_\lambda$ is continuous.*

Proof. The map $(\mathscr{C}, \| \quad \|) \to (\mathscr{C}, \| \quad \|)$ defined by $f \to f^{-1}$ is a homeomorphism. This with Proposition 3 yields the result. Q.E.D.

Let τ be equal to $\bar{\tau}$ defined in the above example. Then we have the following result: as $\lambda \to 2$, the continuous ergodic measures m_λ, invariant with respect to $\bar{\tau}$, approach the absolutely continuous ergodic measure m, invariant under $\bar{\tau}$, uniformly.

For the same example, let

$$\eta_\lambda(A) = m(\psi_\lambda(A))$$

denote the continuous ergodic measure, invariant under τ_λ, induced by $\bar{\tau}$. Reasoning as above, it follows that $\lambda_n \to \lambda$ implies $\eta_{\lambda_n}(0,x) \to \eta_\lambda(0,x)$ uniformly. This result is similar to results obtained in [3] for absolutely continuous invariant measures.

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